INDIAN INSTITUTE OF TECHNOLOGY JODHPUR
Mathematics-II (12001)
Tutorial Sheet 1

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1. Show that for any $m \times n$ matrix $A = \{a_{ij}\}, A = 0$ if and only if $Tr(A^tA) = 0 (or A^tA = 0)$. $Tr(A)$ is the sum of diagonal elements of A, called as trace of A.

2. For any $m \times n$ matrix A and $n \times p$ matrices B and C, show that $AB = AC$ if and only if $A^tAB = A^tAC$.

3. An $n \times n$ matrix $P = \{p_{ij}\}$ is called a stochastic matrix if each of its rows is a probability vector. i.e., If each entry of P is nonnegative and the sum of the entries of each row is 1. Let A and B two stochastic matrices of order $n \times n$. Then the product matrix AB is also a stochastic matrix?

4. A matrix P of order $n \times n$ is called projection matrix if and only if $P^t = P$ and $P^2 = P$.

   (i) Suppose P_1 and P_2 are projection matrices. Then $P_1 + P_2$ and P_1P_2 are projection matrices?

   (ii) Let P be a projection matrix and I_n be a identity matrix of order n. Is $I_n - P$ a projection matrix? Calculate $P(I_n - P)$.

   (iii) Suppose $X \in \mathbb{R}^n$ be a non zero column vector. Define $A = \frac{XX^t}{X^tX}$. Then prove that A is projection matrix.

5. Let A, X, Y be an $n \times n$ matrices. Assume that $XA = I_n$ and $AY = I_n$. Prove that $X = Y$.

6. Calculate A^n for the given $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the real numbers a, b such that $A^2 + aA + bI_3 = 0$. Show that there are real numbers $c_0, c_1, c_2, \cdots, c_n$ such that

$$A^{-n} = c_0I + c_1A^2c_2A^3 + \cdots + c_nA^n.$$

7. If $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, show that the matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the equation$AX = XB$ if and only if X is a scalar multiple of $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

8. A non-zero matrix A is said to be nilpotent of order k where $k \in \mathbb{N}$, if $A^k = 0$ and $A^n \neq 0$, for all $n < k$. In this case, verify that $I - A$ has the inverse $I + A + A^2 + \cdots + A^{k-1}$.

Show that the matrix $A = \begin{pmatrix} -1 & -5 & 4 \\ 0 & -2 & -1 \\ 1 & 3 & 3 \end{pmatrix}$ is nilpotent of order 3, and hence find the inverse of the matrix $\begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 1 \\ -1 & -3 & -2 \end{pmatrix}$.

9. Let A, B, C, D be $n \times n$ matrices over $\mathbb{R}$. Assume that AB^t and CD^t are symmetric and $AD^t - BC^t = I_n$. Show that $A^tD - C^tB = I_n$.

10. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B, A^3 = B^3$ and $A^2B = B^2A$. Is $A^2 + B^2$ invertible?

11. Prove that for each $n \times n$ matrix A, $A^t.A$ is symmetric.

12. Prove that if matrix A is skew-symmetric, then $A.A$ is symmetric.