Tutorial 4 Solutions

T4.1

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \longrightarrow (1) \text{ (by definition)}$$

$$\langle x^2\rangle = \int \psi^* \hat{x}^2 \psi \, d\tau \qquad \text{Here } \psi = \psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha x^2}{2}}$$

$$\langle x^2\rangle = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} x^2 \, dx$$

$$= 2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \int_{0}^{\infty} e^{-\alpha x^2} x^2 \, dx$$

$$= \frac{1}{2\alpha} = \frac{h}{4\pi \nu m} \longrightarrow (2)$$

$$\langle x\rangle = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} x \, dx = 0 \longrightarrow (3)$$

Hence,

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{h}{8\pi^2 \nu m}} \longrightarrow (4)$$

Similarly,

$$\langle p_x^2\rangle = \frac{h\nu m}{2}$$

$$\langle p_x\rangle = 0$$

$$\Delta p_x = \sqrt{\frac{h\nu m}{2}} \longrightarrow (5)$$

$$\Delta x \, \Delta p_x = \sqrt{\frac{h}{8\pi^2 \nu m} \, \frac{h\nu m}{2}} = \frac{h}{4\pi} \longrightarrow (6)$$

Equ. (6) is the uncertainty Principle.

Taking Δx and Δp_x to be minimum values in x and p_x, we can calculate the minimum energy in $\psi_0(x)$.

$$E_0 = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = \frac{\Delta p_x^2}{2m} + \frac{1}{2}k\,\Delta x^2$$

$$= \frac{1}{2m}\,\frac{h\nu m}{2} + \frac{1}{2}\,4\pi^2\nu^2 m \frac{h}{8\pi^2 \nu m} = \frac{h\nu}{4} + \frac{h\nu}{4}$$

$$\boxed{E_0 = \frac{h\nu}{2}} \rightarrow \text{This is the zero point energy.}$$

T4.2

For an operator to be linear,

$$\hat{A}\,(f(x)+g(x)) = \hat{A} f(x) + g(x)$$

$$\hat{A}\, c f(x) = c\, \hat{A} f(x)$$

$$\hat{\Pi}[f(x)+g(x)] = \hat{\Pi} f(x) + \hat{\Pi}\, g(x)$$

$$f(-x)+g(-x) = f(-x)+g(-x)$$

$$\hat{\Pi}\, c f(x) = c\, \hat{\Pi} f(x)$$

$$c f(-x) = c f(-x)$$

Hermitian

$$\int \psi^*_{(x)} \hat{\Pi}\, \psi(x)\, dx = \int \psi(x) [\hat{\Pi}\, \psi(x)]^* dx$$

$$\int \psi^*_{(x)}\, \psi(-x)\, dx = \int \psi(x)\, [\psi(-x)]^* dx$$

$$\int \psi^*(x)\, \psi(-x)\, dx = \int \psi(x)\, \psi^*(-x)\, dx$$

Hence Parity operator is linear Hermitian.

T4.3

$$\phi = \int \psi_{v_1}(x)\; x\; \psi_{v_2}(x)\, dx$$

ϕ will be nonzero when v_1 is even and v_2 is odd or v_1 is odd and v_2 is even.

$$\phi = \int_{-\infty}^{+\infty} (\text{even})\,(\text{odd})\,(\text{odd}) = \int_{+\infty}^{+\infty} \text{even} \neq 0$$

$$\phi = \int_{-\infty}^{+\infty} (\text{odd})\,(\text{odd})\,(\text{even}) = \int_{-\infty}^{+\infty} \text{even} \neq 0$$

ϕ will be zero when v_1 and v_2 are either even or odd.

$$\phi = \int_{-\infty}^{+\infty} (\text{even})\,(\text{odd})\,(\text{even}) = \int_{-\infty}^{+\infty} (\text{odd}) = 0$$

$$\phi = \int_{-\infty}^{+\infty} (\text{odd})\,(\text{odd})\,(\text{odd}) = \int_{-\infty}^{+\infty} (\text{odd}) = 0$$

[illegible] either one and the other one odd

T4.4

$$\int \psi^* \hat{A} \psi \, d\tau = -\int \psi (\hat{A}\psi)^* d\tau$$

Let $\hat{A}\psi = a\psi$

$$\int \psi^* a \psi \, d\tau = -\int \psi (a\psi)^* d\tau$$

$$a \int \psi^* \psi \, d\tau = -a^* \int \psi \psi^* d\tau$$

$$\boxed{a = -a^*} \longrightarrow 1$$

Equ. 1 will be satisfied only when a is purely imaginary.

Let $a = x + iy$

$a \neq a^*$

$x + iy \neq x - iy$

Let $a = iy$

$a = a^*$

$iy = -(iy)^* = iy$