Tutorial #6 solutions

T6.1

(a) $|\alpha\rangle = c_1|a_1\rangle + c_2|a_2\rangle + c_3|a_3\rangle$

$$\langle\alpha|\alpha\rangle = \left[c_1^*\langle a_1| + c_2^*\langle a_2| + c_3^*\langle a_3|\right]\left[c_1|a_1\rangle + c_2|a_2\rangle + c_3|a_3\rangle\right]$$

$$= c_1^*c_1\langle a_1|a_1\rangle + c_1^*c_2\langle a_1|a_2\rangle + c_1^*c_3\langle a_1|a_3\rangle + c_2^*c_1\langle a_2|a_1\rangle + c_2^*c_2\langle a_2|a_2\rangle$$
$$+ c_2^*c_3\langle a_2|a_3\rangle + c_3^*c_1\langle a_3|a_1\rangle + c_3^*c_2\langle a_3|a_2\rangle + c_3^*c_3\langle a_3|a_3\rangle$$

$$= |c_1|^2 + |c_2|^2 + |c_3|^2$$

$$= 0.447^2 + 0.316^2 + 0.5^2 + 0.316^2 + 0.387^2 + 0.448^2$$

$$\boxed{\langle\alpha|\alpha\rangle = 0.999}$$

(b) $\langle\alpha| = c_1^*\langle a_1| + c_2^*\langle a_2| + c_3^*\langle a_3|$

$$c_1^* = 0.447 - 0.316i$$
$$c_2^* = 0.5 + 0.316i$$
$$c_3^* = 0.387 - 0.448i$$

(c)

$$\sum_{a'} |a'\rangle\langle a'| = 1$$
$$|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + |a_3\rangle\langle a_3| = 1$$

(d)

$$\hat{H} \doteq \begin{pmatrix} \langle a_1|H|a_1\rangle & \langle a_1|H|a_2\rangle & \langle a_1|H|a_3\rangle \\ \langle a_2|H|a_1\rangle & \langle a_2|H|a_2\rangle & \langle a_2|H|a_3\rangle \\ \langle a_3|H|a_1\rangle & \langle a_3|H|a_2\rangle & \langle a_3|H|a_3\rangle \end{pmatrix}_{3\times3}$$

$$|\alpha\rangle \doteq \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \langle a_3|\alpha\rangle \end{pmatrix}_{3\times1}$$

T6.2

$$\langle \alpha | \hat{A}\beta \rangle = \langle \hat{A}\alpha | \beta \rangle \implies \int \psi_\alpha^* \hat{A}\,\psi_\beta\, d\tau = \int \psi_\beta (\hat{A}\psi_\alpha)^* d\tau$$

(a) For arbitrary $|\alpha\rangle$ and $|\beta\rangle$

$$\langle \alpha | (\hat{H}+\hat{K}) | \beta \rangle = \langle \alpha | \hat{H} | \beta \rangle + \langle \alpha | \hat{K} | \beta \rangle$$

$$= \langle \hat{H}\alpha | \beta \rangle + \langle \hat{K}\alpha | \beta \rangle$$

$$= \langle (\hat{H}+\hat{K})\alpha | \beta \rangle$$

(b) $\langle \alpha | c\hat{A}\beta \rangle = c\langle \alpha | \hat{A} | \beta \rangle \longrightarrow (1)$

$\langle c\hat{A}\alpha | \beta \rangle = c^* \langle \hat{A}\alpha | \beta \rangle \longrightarrow (2)$

From (1) and (2), $c\hat{A}$ will be Hermitian when c is real i.e., $c = c^*$

(c) $\langle \alpha | \hat{H}\hat{K} | \beta \rangle = \langle \hat{H}\alpha | \hat{K}\beta \rangle$

$= \langle \hat{K}\hat{H}\alpha | \beta \rangle$

Hence $\hat{H}\hat{K}$ is Hermitian when $\hat{H}$ and $\hat{K}$ commute i.e., $\hat{H}\hat{K} = \hat{K}\hat{H}$

T6.3

$$\frac{d^2}{dx^2} e^x = (1) e^x$$

$$\frac{d^2}{dx^2} e^{-x} = (-1)(-1) e^{-x} = (1) e^{-x}$$

e^x and e^{-x} are degenerate with eigenvalue 1. Hence we can construct linear Combination of e^x and e^{-x} to form new functions which will also be eigenfunctions of the system.

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

Further, $\sinh x$ and $\cosh x$ are also orthogonal.
