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Achieving ergodicity in quasi-static MIMO with polynomial-time complexity and one bit of feedback

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***Abstract*—This study establishes the computational-complexity savings that a properly positioned single bit of feedback can provide in the computationally intense setting of quasi-static MIMO communications. Specifically, the work identifies novel practically constructed feedback schemes and explicit and non-random multiple-input multiple-output (MIMO) encoding-decoding schemes that, in the presence of a single bit of feedback, jointly guarantee the optimal diversity-multiplexing tradeoff (DMT) with a polynomial time complexity. Going one step further, the work also presents an opportunistic communication scheme that, at all rates including rates close to the maximum multiplexing gain, can provide near-ergodic reliability at just polynomial time computational complexity costs. This is the best known computational complexity that suffices to achieve near-ergodic reliability in the quasi-static MIMO settings.**

***Index Terms*—Computational complexity, detection, lattice code design, MIMO, diversity-multiplexing gain tradeoff.**

I. Introduction

A. System Model

This work considers an $n_T \times n_R$ quasi-static multiple-input multiple-output (MIMO) channel model given by

$$\mathbf{Y}_C = \theta \mathbf{H}_C \mathbf{X}_C + \mathbf{W}_C, \tag{1}$$

where $\mathbf{X}_C \in \mathbb{C}^{n_T \times T}$, $\mathbf{Y}_C \in \mathbb{C}^{n_R \times T}$ and $\mathbf{H}_C \in \mathbb{C}^{n_R \times n_T}$ denote transmitted codeword matrix, received signal matrix and channel matrix with entries from i.i.d. fading statistics respectively, and where the scaling factor θ is chosen such that $\mathbb{E}(\|\theta \mathbf{X}_C\|^2) \leq \rho T$, where ρ denotes signal-to-noise ratio (SNR). We finally consider the rate $R = \frac{1}{T} \log |\mathcal{X}_C|$ in bits per channel use (bpcu), where $|\mathcal{X}|$ denotes the cardinality of $\mathcal{X}$.

The rate-reliability performance of quasi-static MIMO can be characterized using the diversity multiplexing tradeoff (DMT, cf. [1]) that describes the relationship between the rate R and the probability of error P_{err} using the high SNR measures of multiplexing gain $r := R/\log \rho$ and diversity gain $d(r) := -\lim_{\rho \to \infty} \frac{\log P_{\text{err}}}{\log \rho}$. The same work in [1] also revealed the optimal DMT, for the case of no feedback, as the maximum possible diversity gain $d^*(r)$ for a given r.

B. Background and Previous Work

In quasi-static MIMO communications, rate-reliability and encoding-decoding computational complexity are widely considered to be limiting and interrelated bottlenecks. For this reason, any attempt to significantly reduce complexity may be at the expense of a substantial degradation in error-performance (cf. [2], [3]). Finding out computationally efficient decoding algorithms that allow for near-optimal behavior with reduced complexity cost remains an important research topic of substantial practical interest ([4]–[14]). In particular, recently substantial amount of work have focused on identifying explicit and non-random MIMO encoding-decoding schemes that achieve the optimal or near-optimal error performance with computationally efficient receivers (cf. [3], [11]–[14]). Specifically, the work in [3], [13] presented computationally efficient maximum likelihood (ML)-based sphere decoding solutions that employ fixed search radius sphere decoder (SD) and time-out policies to achieve arbitrary close to brute-force ML performance with substantially reduced complexity costs. The achieved computational savings vary with the desired DMT (cf. [13, Theorem 2]) and reveal an exponential reduction in the required computational resources, however, the decoding complexity still grows exponentially in the number of codeword bits, transmission rate and system dimensionality.

Another computationally efficient decoder for lattice designs is lattice decoder that has been studied extensively in [4], [6], [7], [11], [12]. The work in [12] established equivalence of ML-based and lattice based sphere decoding solutions in terms of the complexity costs and also revealed that for large MIMO systems these computational costs can be prohibitively large and render system implementation infeasible, bringing to the fore the need for methods that manage to achieve the same near-optimal performance, but do so with much reduced computational resources. The same work in [12] also presented a computationally efficient lattice reduction (LR)-aided MMSE-preprocessing lattice decoder that allows for the optimal diversity-multiplexing behavior ($d^*(r)$) with computational resources of the order of ρ^x for $x > 0$. The work in [11] presented LR-aided linear decoder that achieves ($d^*(r)$) with computational resources of the order of $O(\log \rho)$, which actually is the order of the complexity cost associated with LLL-based LR. It is the case though that such LR-aided methods cannot be readily applied to many communication scenarios including very large MIMO systems (cf. [11]).

Motivated by the considerable magnitude of the complexity cost of SD based methods (cf. [3], [13]) and non-feasibility of LR-aided methods, the work in [14] showed that if the feedback is used for reducing complexity, rather than in improving reliability as shown in [15], then a properly positioned single bit of feedback can provide exponential reductions in the complexity costs by bringing down complexity costs of achieving $d^*(r)$ from being exponential in the number of codeword bits (cf. [13]) to being at most exponential in the

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rate. It is the case tough that feedback-aided complexity of [14], albeit significantly smaller than those required in the absence of feedback (cf. [3], [13]), again grows exponentially in the rate, and remain prohibitive for many MIMO scenarios, leaving open the quest for the holy grail of wireless communications, i.e., the search of decoding algorithms achieving optimal diversity-multiplexing behavior with polynomial time complexity.

C. Contributions

This work improves upon the result of [14] and identifies the first practically constructed feedback schemes, as well as simple lattice code designs and decoders, that jointly guarantee $d^*(r)$ with just a polynomial time complexity. Furthermore, the work presents an opportunistic communication scheme that with a properly positioned single bit of feedback achieves an ergodic-like (very high diversity gain for all rates including rates arbitrary close to the maximum multiplexing gain) error performance again at polynomial-time complexity costs. The derived result is the best known computational complexity that suffices to achieve near-ergodic behavior in the quasi-static MIMO settings.

II. Complexity analysis for DMT $d^*(r)$

This section analyzes the complexity saving that can be attained by proper utilization of feedback and establishes the feedback-aided decoding complexity required to achieve the optimal DMT $d^*(r)$. The following holds for the $n_{\rm T} \times n_{\rm R}$ ($n_{\rm R} \geq n_{\rm T}$), i.i.d. regular fading[1] MIMO channel.

Theorem 1: Linear decoding with a properly positioned single-bit of feedback and ARQ signaling achieves DMT $d^*(r)$ with the polynomial time complexity costs.

The proof of the above theorem includes the derivation of the decoding complexity and also the constructive achievement of this rate-reliability-complexity limit. The constructive part of the proof is based on designing ARQ schemes, lattice designs and decoding policies that meet the complexity limit.

The proposed ARQ scheme consists of two rounds, where each message is associated to a unique block $[\mathbf{X}_C^1 \ \mathbf{X}_C^2]$ of signaling matrices, where each $\mathbf{X}_C^i \in \mathbb{C}^{n_T \times T_i}, \ i = 1, 2,$ corresponds to the $n_{\rm T} \times T_i$ matrix of signals sent during the ith round. The accumulated code matrix at the end of the second round, takes the form $\mathbf{X}_C^{ARQ,2} = [\mathbf{X}_C^1 \ \mathbf{X}_C^2] \in \mathbb{C}^{n_T \times (T_1+T_2)}$. We note that the signals $\mathbf{X}_C^{ARQ,2}$ are drawn from a lattice design that ensures unique decodability at every round[2]. The channel remains constant during each block of two-round ARQ signaling. In the quasi-static case of interest, the received signal accumulated at the end of the ℓ-th round takes the form

$$\mathbf{Y}_C^{\ell} = \theta \mathbf{H}_C \mathbf{X}_C^{ARQ,\ell} + \mathbf{W}_C^{\ell}, \quad \ell = 1, 2, \tag{2}$$

[1]The i.i.d. regular fading statistics satisfy the general set of conditions as described in [16], where a) the near-zero behavior of the fading coefficients h is bounded in probability as $c_1|h|^t \leq p(h) \leq c_2|h|^t$ for some positive and finite c_1, c_2 and t, where b) the tail behavior of h is bounded in probability as $p(h) \leq c_2 e^{-b|h|^{\beta}}$ for some positive and finite c_2, b and β, and where c) $p(h)$ is upper bounded by a constant K.

[2]Loosely speaking, unique decodability means that, for any $\ell = 1, 2$, the corresponding $\mathbf{X}_C^{ARQ,\ell}$ carries all bits of information.

where the scaling factor θ is chosen such that $\mathbb{E}(\|\theta \mathbf{X}_C^i\|^2) \leq \rho T_i, \ 1 \leq i \leq 2$.

Another important aspect in ARQ schemes is knowing when to decode and when not to decode across the different rounds and incremental redundancy ARQ lattice design to be used for the signaling. Towards this we have the following definitions.

Definition 1 (Decoding policies): We define *decoding policies* to be the family of policies that perform first round decoding if and only if channel is really good and halt decoding in the first round whenever the minimum singular value of the channel scales as $\rho^{-\epsilon}$ for some $\epsilon > 0$, i.e., $|\sigma_{min}(\mathbf{H}_C^H \mathbf{H}_C)| \leq \rho^{-\epsilon}$ for some $\epsilon > 0$ and which decode at the second round if and only if there is no decoding in the first round.

Definition 2 (ARQ lattice design): We define *ARQ lattice design* to be the $n_{\rm T} \times (n_{\rm T}^2 + 1)$ incremental redundancy lattice code designs that transmits $n_{\rm T} \times 1$ uncoded-QAM symbols in the first round and a $n_{\rm T} \times n_{\rm T}^2$ orthogonal design with rate-$\frac{1}{n_{\rm T}}$ in the second round. The mentioned codes take the simple form

$$\mathbf{X}_C^{ARQ,2} = \begin{bmatrix} f_1 & f_1 & 0 & \cdots & 0 & f_2 & \cdots & 0 \\ f_2 & 0 & f_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n_{\rm T}} & 0 & 0 & \cdots & f_1 & 0 & \cdots & f_{n_{\rm T}} \end{bmatrix}, \tag{3}$$

where $f_i, \ i = 1, \cdots, n_{\rm T}$ belong to the QAM constellation. It is clear that first round duration is $T_1 = 1$ and second round duration is $T_2 = n_{\rm T}^2$.

The proof of Theorem 1 is presented in Appendix A.

Theorem 1 established the complexity savings that feedback provides for a given fixed rate-reliability performance $d^*(r)$. In the following section we establish the complexity costs of achieving full rate-reliability benefits of feedback.

III. Ergodic-like behavior in quasi-static MIMO

The rate-reliability gains of feedback were studied extensively in [15], [17]–[20] where it was shown that ARQ/CSIT feedback can achieve a much higher DMT as compared to $d^*(r)$. In the following work we analyze the complexity costs of achieving the full rate-reliability benefits of feedback. The following holds for the $n_{\rm T} \times n_{\rm R}$ ($n_{\rm R} \geq n_{\rm T}$), i.i.d. regular fading MIMO channel.

Theorem 2: Linear decoding with a properly positioned single-bit of feedback and opportunistic signaling achieves ergodic-like behavior in quasi-static MIMO settings with polynomial time complexity costs.

The proof of the above theorem is again based on providing lattice designs and communication schemes that allow to achieve ergodic-like behavior with polynomial time complexity. An important aspect of an opportunistic communication scheme is knowing when to transmit and when not to transmit across the different channel realizations. Towards this we have the following definition.

Definition 3 (Communication windows): We define *communication windows* to be the instances of channel realizations

whenever the minimum singular value of the channel scales $|\sigma_{min}(\mathbf{H}_C^H\mathbf{H}_C)| > \rho^{-\epsilon}$ for some $\epsilon > 0$.
The single bit of CSIT feedback provides transmitter with the requisite information regarding communication windows. The transmitter transmits independent uncoded QAM symbols form each transmit antenna (V-BLAST) and the receiver performs ZF decoding. The proof of Theorem 2 is presented in Appendix B.

IV. Conclusions

In the setting of quasi-static MIMO, we have shown how specific encoding-decoding policies and a properly positioned single bit of ARQ feedback can achieve $d^*(r)$ at polynomial time computational complexity. We also provided a concise characterization of the best known computational complexity that suffices to achieve near-ergodic reliability in the quasi-static MIMO settings. The presented schemes can serve as an alternative to rate-adaptation techniques which require considerable CSIT to guarantee similar error performance.

Appendix A
Proof of Theorem 1

The proof includes the derivation of the complexity cost and also the constructive achievement of this complexity limit with the proposed ARQ scheme.

A. Complexity Analysis

The complexity costs of both the first round ZF decoder and the second round linear decoder for orthogonal design are of the order of $O(n_{\mathrm{T}}^2)$, resulting in the overall complexity costs that are polynomial time complexity.

For the proof to be complete we must now prove that the aforementioned family of ARQ schemes, decoding policies and lattice designs can indeed achieve the desired DMT $d^*(r)$ with first round ZF decoding and second round linear decoding.

B. Achievability

To prove DMT optimality for ARQ scheme it is sufficient to show

- *Condition 1:* that with high probability there will be just a single ARQ round, i.e., the probability of NACK event $(\overline{\mathcal{A}}_1)$ for first round is $\mathrm{P}\left(\overline{\mathcal{A}}_1\right) \doteq \rho^{-\mathcal{T}}$, for[3] some $\mathcal{T} > 0 \;\forall\; 0 \leq r \leq n_T$,
- *Condition 2:* the error probability of the first round ZF decoding the ST code $\mathbf{X}_C^1$ is no larger than that incurred by the linear decoder applied to the task of decoding the ST code $\mathbf{X}_C^2$, i.e., $\mathrm{P}\left(r\right)_{err}^{ARQ,1} \dot{<} \mathrm{P}\left(r_2\right)_{err}^{ARQ,2}$, and
- *Condition 3:* the orthogonal lattice design $\mathbf{X}_C^2$ in second round achieves diversity gain $d_{ARQ,2}(r_2) \geq d^*(r)$.

Condition 1: Towards proving the first condition above, we recall that the decoder of first round sends NACK if and only if $\sigma_1(\mathbf{H}_C^H\mathbf{H}_C) \leq \rho^{-\epsilon}$ for some $\epsilon > 0$ and ACK otherwise, where $\sigma_1(\mathbf{H}_C^H\mathbf{H}_C) \leq \cdots \leq \sigma_{n_{\mathrm{T}}}(\mathbf{H}_C^H\mathbf{H}_C)$ are the singular values of $\mathbf{H}_C^H\mathbf{H}_C$. Thus, the probability of a NACK being received at the end of the first round is given by

$$\mathrm{P}\left(\overline{\mathcal{A}}_1\right) = \mathrm{P}\left(\sigma_1(\mathbf{H}_C^H\mathbf{H}_C) \leq \rho^{-\epsilon}\right).$$

For i.i.d. regular fading channel $\mathbf{H}_C$, from [16] it follows that

$$\mathrm{P}\left(\sigma_1(\mathbf{H}_C^H\mathbf{H}_C) \leq \rho^{-\epsilon}\right) \dot{\leq} \int_{\mathcal{A}} \rho^{-I(\boldsymbol{\mu})} \doteq \rho^{-I(\boldsymbol{\mu}^*)},$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_{n_T})$, where $\mu_i \triangleq -\frac{\log \sigma_i(\mathbf{H}_C^H\mathbf{H}_C)}{\log \rho}$, $i = 1, \cdots, n_{\mathrm{T}}$, where asymptotic equality follows from Varadhan's lemma [21], where[4]

$$I(\boldsymbol{\mu}) = \sum_{j=1}^{n_T}(n_R - n_T + 2j - 1)\mu_i + \frac{n_R n_T t}{2}\mu_{n_T},$$

$$\mathcal{A} = \{\boldsymbol{\mu} \mid \mu_1 \geq \epsilon, \mu_i \geq 0, \text{ for } i = 2, \cdots, n_T\},$$

and where $\boldsymbol{\mu}^* = \arg\inf_{\mathcal{A}} I(\boldsymbol{\mu})$. It follows that $I(\boldsymbol{\mu}^*) = (n_R - n_T + 1)\epsilon$ and consequently, we have that for $0 \leq r_1 \leq n_T$

$$\mathrm{P}\left(\overline{\mathcal{A}}_1\right) \dot{\leq} \rho^{-(n_R - n_T + 1)\epsilon}.$$

It is clear that the proposed ARQ scheme achieves a multiplexing gain value of r that is given by

$$r = r_1 \mathrm{P}\left(\mathcal{A}_1\right) + r_2 \mathrm{P}\left(\overline{\mathcal{A}}_1\right) = r_1(1 - \rho^{-\mathcal{T}}) + r_2\rho^{-\mathcal{T}},$$

where r_1 and r_2 denote multiplexing gain values for the first and the second round respectively and where $\mathcal{T} := (n_R - n_T + 1)\epsilon$. In the high SNR regime, we have that

$$\lim_{\rho\to\infty} r = r_1 = r_2(n_{\mathrm{T}}^2 + 1). \tag{4}$$

It is in the proof of this condition that we make use of the fact that communication takes place over i.i.d. regular fading statistics, rest of the proof holds irrespective of the fading statistics.

Condition 2: For second condition we need to show that $\mathrm{P}\left(r\right)_{err}^{ARQ,1} \dot{\leq} \mathrm{P}\left(r_2\right)_{err}^{ARQ,2}$. Towards proving this condition we evaluate $\mathrm{P}\left(r\right)_{err}^{ARQ,1}$ by considering the first round system model given by

$$\mathbf{y}_C = \theta \boldsymbol{H}_C \boldsymbol{x}_C + \boldsymbol{w}_C,$$

where $\mathbf{y}_C = \mathbf{Y}_C^1$, where $\boldsymbol{x}_C = \mathbf{X}_C^1$, where $\boldsymbol{w}_C = \mathbf{W}_C^1$ and where $\theta^2 = \rho^{1-\frac{r}{n_{\mathrm{T}}}}$. For ZF decoder decision step consists of mapping each element of the ZF filter (F_{ZF}) output vector onto an element of the symbol alphabet, i.e.,

$$\hat{\boldsymbol{x}} = f(F_{ZF}.\, \mathbf{y}),$$

where $F_{ZF} = \frac{1}{\theta}(\mathbf{H}_C^H\mathbf{H}_C)^{-1}\mathbf{H}_C^H$ and where $f(\bullet)$ denotes quantization function that maps vector $(\bullet)$ onto an element of the symbol alphabet. By substituting for F_{ZF} and $\mathbf{y}$ we can further simplify the decision metric as

$$\begin{aligned}\hat{\boldsymbol{x}} &= f(\frac{1}{\theta}(\mathbf{H}_C^H\mathbf{H}_C)^{-1}\mathbf{H}_C^H(\theta \boldsymbol{H}_C\boldsymbol{x}_C + \boldsymbol{w}_C)),\\ &= f(\boldsymbol{x}_C + \frac{1}{\theta}(\mathbf{H}_C^H\mathbf{H}_C)^{-1}\mathbf{H}_C^H\boldsymbol{w}_C).\end{aligned}$$

[3] We use $\doteq$ to denote the *exponential equality*, i.e., we write $f(\rho) \doteq \rho^B$ to denote $\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho} = B$, and $\dot{\leq}, \dot{\geq}$ are similarly defined.

[4] Recall that parameter t was introduced as a parameter that regulates the near zero behavior of the random variable for i.i.d. regular fading.

It is clear that ZF decoder makes an error if $\|\frac{1}{\theta}(\mathbf{H}_C^H\mathbf{H}_C)^{-1}\mathbf{H}_C^H\boldsymbol{w}_C\|^2 > d^2$, where $2d$ (independent of ρ) is the minimum euclidean distance for QAM constellation. Thus, the probability of error for the first round ZF decoding is given by

$$\begin{aligned}
\mathrm{P}\,(r)_{err}^{ARQ,1} =& \mathrm{P}\left(\|\frac{1}{\theta}(\mathbf{H}_C^H\mathbf{H}_C)^{-1}\mathbf{H}_C^H\boldsymbol{w}_C\|^2 > d^2\right),\\
\leq& \mathrm{P}\left(\frac{1}{\theta^2}\sigma_{max}^2((\mathbf{H}_C^H\mathbf{H}_C)^{-1})\|\mathbf{H}_C^H\boldsymbol{w}_C\|^2 > d^2\right),\\
\overset{(a)}{=}& \mathrm{P}\left(\|\mathbf{H}_C^H\boldsymbol{w}_C\|^2 > d^2\theta^2\sigma_1^2(\mathbf{H}_C^H\mathbf{H}_C)\right),\\
\overset{(b)}{\leq}& \mathrm{P}\left(\|\mathbf{H}_C^H\boldsymbol{w}_C\|^2 > d^2\rho^{1-\frac{r}{n_\mathrm{T}}-2\epsilon}\right),\\
\doteq& \mathrm{P}\left(\|\boldsymbol{w}_C\|^2 > \rho^{1-\frac{r}{n_\mathrm{T}}-2\epsilon}\right),\\
\overset{(c)}{\dot{\leq}}& \mathrm{P}\left(\|\boldsymbol{w}_C\|^2 > z\log\rho\right),\\
\overset{(d)}{\dot{<}}& \rho^{-z_1},
\end{aligned} \tag{5}$$

where (a) follows from the fact that $\sigma_{max}((\mathbf{H}_C^H\mathbf{H}_C)^{-1}) = \frac{1}{\sigma_1(\mathbf{H}_C^H\mathbf{H}_C)}$, where (b) follows from the fact that first round decoding is performed if and only if $\sigma_1(\mathbf{H}_C^H\mathbf{H}_C) > \rho^{-\epsilon}$ and where (c) follows from the fact that for sufficiently small ϵ, $1 - \frac{r}{n_T} - 2\epsilon > 0$ for $0 \leq r < n_T$ which in turn implies that $z\log\rho \dot{<} \rho^{1-\frac{r}{n_\mathrm{T}}-2\epsilon}$ $\forall$ $0 < z < \infty$ (z is independent of ρ) and where (d) follows for any $0 < z_1 < z$. For $z_1 = d_{ARQ,2}(r_2)$, the error probability of (5) implies that

$$\mathrm{P}\,(r)_{err}^{ARQ,1} \dot{<} \mathrm{P}\,(r_2)_{err}^{ARQ,2}. \tag{6}$$

Condition 3: To satisfy third condition we need to prove that $d_{ARQ,2}(r_2) \geq d^*(r)$. For the $n_\mathrm{T} \times n_\mathrm{T}^2$ orthogonal design $\mathbf{X}_C^2$ with rate-$\frac{1}{n_\mathrm{T}}$, it is straight forward to show that with linear decoding $\mathbf{X}_C^2$ can achieve a diversity gain of

$$d(\tilde{r}) = n_\mathrm{R}n_\mathrm{T}\,(1 - n_\mathrm{T}\tilde{r})\;\;\forall\; 0 \leq \tilde{r} \leq \frac{1}{n_T},$$

where $\tilde{r}$ denotes multiplexing gain of code $\mathbf{X}_C^2$. We know that $\tilde{r} = \frac{r_2(n_\mathrm{T}^2+1)}{n_\mathrm{T}^2}$, as a result we get that

$$d_{ARQ,2}(r_2) = n_\mathrm{R}n_\mathrm{T}\left(1 - \frac{r_2(n_\mathrm{T}^2+1)}{n_\mathrm{T}}\right).$$

From (4) we have that $r = r_2(n_\mathrm{T}^2+1)$, making this substitution we get that

$$d_{ARQ,2}(r_2) = n_\mathrm{R}n_\mathrm{T}\left(1 - \frac{r}{n_\mathrm{T}}\right) \geq d^*(r). \tag{7}$$

We have shown that the proposed ARQ scheme achieves the desired DMT $d^*(r)$ with polynomial-time complexity. This proves Theorem 1. □

Appendix B
Proof of Theorem 2

Regarding complexity costs, ZF decoding is known to introduce only polynomial time complexity and the proof of DMT follows directly from (5) of the proof of Theorem 1. In high SNR settings, the resulting diversity gain $d(r_e) \to \infty$ (as $z_1 \to \infty$) for an average communication rate denoted by multiplexing gain $r_e = r_1(1 - \rho^{-\mathcal{T}})$ with $\mathcal{T} > 0$ for all $0 \leq r_e < n_\mathrm{T}$. This proves Theorem 2. □

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