Complexity Theory
Problem Set 2

1. Show that if P = NP, then every language $L \in$ NP, except $L = \phi$ and $L = \Sigma^*$, is **NP-complete**.

2. Show that if *FACTOR* is **NP-complete**, then **NP** = **coNP**.

3. Suppose $L_1, L_2 \in$ NP $\cap$ coNP. Then show that $L_1 \oplus L_2$ is in NP $\cap$ coNP, where $L_1 \oplus L_2 = \{x \mid x \text{ is in exactly one of } L_1, L_2\}$.

4. Prove that the following languages are **NP-complete**.

a) 0/1-*INTEGERPROG* = $\{x \mid x$ is a list of m linear inequalities with rational coefficients over n variables $u_1, u_2, \ldots u_n$ (a linear inequality has the form $a_1u_1 + a_2u_2 + \ldots + a_nu_n \leq b$ for some coefficients $a_1, \ldots, a_n, b$) such that there is an assignment of 0s and 1s to $u_1, u_2, \ldots u_n$ satisfying all the inequalities$\}$

b) *CLIQUE* = $\{(G, k) \mid G$ has a clique of at least k many vertices$\}$

c) *EXACTONE3SAT* = $\{\phi \mid \phi$ is a 3CNF formula such that there exists a satisfying assignment u for ϕ such that every clause of ϕ has exactly one True literal$\}$

d) *SUBSETSUM* = $\{(S, k) \mid S$ is a set of n numbers such that there is a subset of S whose sum of elements is $k\}$

e) *UHAMPATH* = $\{G \mid G$ is an undirected graph that contains a hamiltonian path.$\}$

5. Show that *HALT* is **NP-hard**. Is it **NP-complete**?

6. Show that *MULT* = $\{(\langle n \rangle, \langle m \rangle, \langle nm \rangle) \mid n, m \in \mathbb{N}\}$ is in **L**.

Solutions

1. Let L' be any language in **NP**. We will show that L' can be reduced to any language L in **NP** if $L \neq \phi$ and $L \neq \Sigma^*$ and **P** = **NP**. Consider the function f such that $f(x) = a$ when $x \in L'$ and $f(x) = b$ when $x \notin L'$, such that $a \in L$ and $b \notin L$. This function f is also polynomial-time computable because if **P** = **NP**, then given x you can find whether $x \in L'$ in polynomial-time and then map it to a fixed $a \in L$ or a fixed $b \notin L$ appropriately.

Such an f does not exist for $L = \phi$ and $L = \Sigma^*$ because $L = \phi$ does not have an $a \in L$ and $L = \Sigma^*$ does not have a $b \notin L$.

2. We will show both **NP** ⊆ **coNP** and **coNP** ⊆ **NP**, if *FACTOR* is **NP-complete**.

FACTOR ∈ **NP-complete** implies **NP** ⊆ **coNP**. Take any $L \in$ **NP**, then **coNP** NTM M for L on input x will first reduce x to $f(x)$, such that $x \in L \iff f(x) \in$ *FACTOR*, and then run **coNP** NTM of *FACTOR* on $f(x)$. If $x \in L$, then M will accept it along all paths, else M will reject it along at least one path.

Now we can use **NP** ⊆ **coNP** to prove that **coNP** ⊆ **NP**. **NP** ⊆ **coNP** $\implies$ *SAT* ∈ **coNP** $\implies$ $\overline{SAT}$ ∈ **NP** $\implies$ **coNP** ⊆ **NP** ($\because$ $\overline{SAT}$ is **coNP-complete**).

3. See the answer to Problem 5 here
https://courses.engr.illinois.edu/cs579/sp2017/solutions/hw1sol.pdf

4. a) See Theorem 2.16 in Arora-Barak.
b) Reduce the Independent Set problem to this. The reduction is (G, k) to (G', k), where G' is a complement of G.
c) See this: https://en.wikipedia.org/wiki/Boolean_satisfiability_problem#Exactly-1_3-satisfiability
d) See this: https://www.cs.mcgill.ca/~lyepre/pdf/assignment2-solutions/subsetSumNPCompleteness.pdf
e) See this: https://www.andrew.cmu.edu/user/ko/pdfs/lecture-21.pdf

5. *HALT* is not **NP-complete** as it would imply that it is in **NP** and hence can be decided by an **EXP** machine. But it is **NP-hard**. See the first answer here.
https://cs.stackexchange.com/questions/69448/show-that-halting-problem-mathsfhp-text-is-mathsfnp-text-hard

6. See the first answer here https://cs.stackexchange.com/questions/87716/how-to-show-mult-abc-a-b-c-binary-natural-numbers-and-a-b-c-is-in-log-s