Complexity Theory
Problem Set 3

1. Show that the following languages are **NL-complete**.

a) *CYCLE* = $\{\langle G \rangle \mid G$ is a directed graph that contains a directed cycle$\}$

b) *STRONGLY-CONNECTED* = $\{\langle G \rangle \mid G$ is a strongly connected directed graph$\}$

c) *2SAT*

2. Show that $\overline{BIPARTITE} \leq_l UPATH$, where

BIPARTITE = $\{G \mid G$ is an undirected bipartite graph$\}$

UPATH = $\{(G, s, t) \mid G$ is an undirected graph and $s, t \in V(G)$ such that $\exists$ a path from s to $t\}$

3. Prove that SPACE(n) $\neq$ **NP**.

4. Define **polyL** to be $\cup_{c>0}$DSPACE($\log^c n$). Steve's class, **SC**, is defined to be the set of languages that can be decided by deterministic TMs that run in polytime and $O(\log^c n)$ space for some $c > 0$. Does Savitch's theorem imply **NL** $\subseteq$ **SC**? Is **SC** = **polyL** $\cap$ **P**?

5. The Japanese game Go-Moku is played by two players, "X" and "O" on a 19 × 19 grid. Players take turns placing markers, and the first player to achieve five of her markers consecutively in a row, column, or diagonal is the winner. Consider this game generalized to an $n \times n$ board.

Let *GM* = $\{\langle B \rangle \mid B$ is a position in generalized Go-Moku, where player "X" has a winning strategy$\}$.

By a position we mean a board with markers placed on it, such as may occur in the middle of a play of the game, together with an indication of which player moves next. Show that *GM* $\in$ **PSPACE**.

6. Show that there exists a computable function $> n$ that is not time-constructible.

7. Show that the following language is undecidable.

$$\{\alpha \mid M_\alpha \text{ is a machine that runs in at most } 100n^2 + 200 \text{ steps.}\}$$

Solutions

1. a) *https://cs.stackexchange.com/questions/109972/showing-cycle-is-nl-complete*

b) Solution 3 here *http://users.cms.caltech.edu/~umans/cs151/soln2.pdf*

c) For **NL** part see this: *https://cs.brown.edu/people/jsavage/book/Page.363.364.pdf*.
For **NL-hard** part see this:
https://cse.iitkgp.ac.in/~abhij/course/theory/CC/Spring04/soln3.pdf.

2. *http://homepage.divms.uiowa.edu/~sriram/131/spring07/homework4Hints.pdf*

3. *https://axion004.wordpress.com/2019/02/22/np-spacen/*

4. It's isn't clear how Savitch's theorem implies $\mathsf{NL} \subseteq \mathsf{SC}$. One way Savitch's theorem would have proved $\mathsf{NL} \subseteq \mathsf{SC}$ if the *REACH* procedure also ran in polynomial time. This is because *PATH* is **NL-hard** and *PATH* can be solved using the *REACH* procedure in $O(\log^2 n)$ space but not in polynomial time.

Also, it is not known whether $\mathsf{SC} = \mathsf{polyL} \cap \mathsf{P}$. Notice that **SC** is the set of languages that can be solved by algorithms/TMs that simultaneously run in polytime and $O(\log^c n)$ space, while $\mathsf{polyL} \cap \mathsf{P}$ is the set of languages that can be solved by algorithms that run in polytime and also by algorithms that run in $O(\log^c n)$ space.

5. See the second problem here *http://web.cse.ohio-state.edu/~rademacher.10/Sp16_6321/ps6sol.pdf*

6. See this *https://math.stackexchange.com/questions/55096/non-time-constructible-functions*

7. See the first answer here: *https://math.stackexchange.com/questions/36704/to-prove-an-undecidable-language-on-halting*