ADSA
Problem Set 5

1. Show that if P = NP, then every problem $L \in$ NP, except $L = \phi$ and $L = \Sigma^*$, is **NP-complete**.

2. Prove the following problems **NP-hard**. You may reduce any known **NP-hard** problem to the below problems to prove them **NP-hard**.

a) *LPath* $= \{\langle G, s, t, k\rangle \mid G$ is a directed graph that contains an st-path of length at least $k\}$

b) *DoubleSat* $= \{\phi \mid \phi$ is a CNF formula with at least two satisfying assignments$\}$.

c) *Clique* $= \{(G, k) \mid G$ has a clique of k many vertices$\}$

d) *ExactOne3SAT* $= \{\phi \mid \phi$ is a 3CNF formula such that there exists a satisfying assignment u for ϕ such that every clause of ϕ has exactly one True literal$\}$

e) *Half-Clique* $= \{\langle G\rangle \mid G$ is an undirected graph having a complete subgraph with at least $n/2$ nodes, where n is the number of nodes in $G\}$.

3. Give an example of a graph for which **Approx-VC** always yields a suboptimal solution.

4. Show how in polynomial time to transform one instance of the TSP into another instance whose cost function satisfies the triangle inequality. The two instances must have the same set of optimal tours. Can such a polynomial time transformation be used to obtain a 2-approximation algorithm for the general TSP problem?

Solutions

Solution 1

Let L be a decision problem in **NP** such that $L \neq \phi$ and $L \neq \Sigma^*$. Let L' be any decision problem in **NP**. We will show that L' can be reduced to L in polynomial-time, thus proving that L is **NP-complete**. Consider a polynomial-time algorithm A such that $A(x) = a$ when $x \in L'$ and $A(x) = b$ when $x \notin L'$, such that $a \in L$ and $b \notin L$. Such an algorithm exists because if **P** = **NP**, then given x it can find whether $x \in L'$ in polynomial-time and then map it to a fixed $a \in L$ or a fixed $b \notin L$ appropriately.

Solution 2

a) Reduce the *DirHamPath* problem to this. The reduction is $\langle G, s, t \rangle$ to $\langle G', s', t', n \rangle$, where $G' = G, s' = s, t' = t$, and n is the number of vertices in G. Now, G contains a Hamiltonian st-path if and only if G' contains a path from s' to t' of length at least n.

b) Reduce *SAT* to *DoubleSAT*. The reduction is ϕ to $\phi' = \phi \wedge (u \vee \neg u)$, where u is an extra variable not in ϕ. Now, if ϕ has a satisfying assignment say z, then ϕ' will have two satisfying assignment $z1$ and $z0$, where 1 and 0 values are given to u, respectively. On the other hand, if ϕ' has at least two satisfying assignments, then they must satisfy clauses in ϕ as well. Therefore, ϕ is also satisfiable.

c) Reduce the *IndSet* problem to *Clique*. The reduction is (G, k) to (G', k), where G' is a complement of G, that is, we put edges between u and v in G' if and only if G does not have an edge between u and v. Proof of correctness is left to you.

d) Read this.

e) Reduce *Clique* to *Half-Clique*. Read this.

Solution 3

Consider a star graph. It has a vertex cover of size 1, but **Approx-VC** will always return a vertex cover of size 2.

Solution 4

Read solution (b) here.